# FAST OPTIMIZATION OF BOOLEAN QUADRATIC FUNCTIONS VIA ITERATIVE SUBMODULAR APPROXIMATION AND MAX-FLOW

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## Introduction:

- **Optimization problem** – minimizing arbitrary quadratic forms over the {0,1} (Boolean) lattice

$$\min_{\mathbf{x}\in\{0,1\}^n}\left\{f(\mathbf{x}) := \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x}\right\}$$

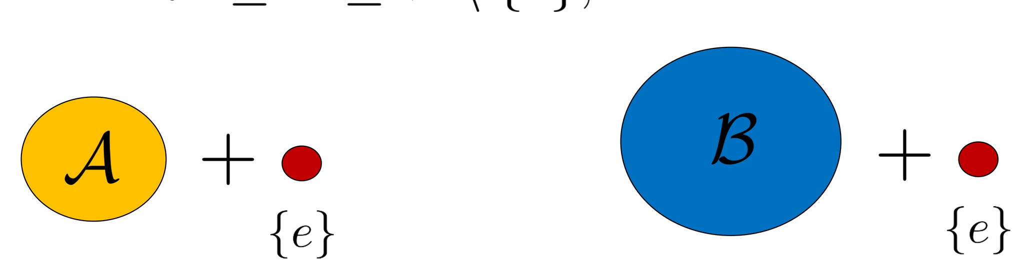
- **Applications**
  i) ML multi-user MIMO detection [Ma *et. al*, 2002]
  ii) Design of CDMA spreading codes [Karystinos-Liavas, 2010]
  iii) Mode computation in {0,1} undirected MRFs [Kolmogorov-Zabih, 2004]
  iv) Mining dense sub-graphs [Tsourakakis *et. al*, 2013]
- **Non-convex and NP-hard**
- **Our approach** – exploit combinatorial structure to construct and iteratively minimize a sequence of global submodular upper bounds on the cost function via the Max-Flow algorithm

## Prior Art:

- **Semi-Definite Relaxation + randomized rounding** – [Luo *et al.*, 2010]. Sub-optimality guarantees in special cases, high complexity
- **Algorithm of** [Karystinos-Liavas, 2010]. Exact maximization of rank-deficient quadratic forms in polynomial-time, high complexity
- **Maximum Flow** – [Kolmogorov-Zabih, 2004]. Exact minimization of quadratic submodular functions in polynomial-time

## Submodularity:

- Given a set of elements $\mathcal{V} = [n] := \{1, \cdots, n\}$, consider a set-function $f : 2^{\mathcal{V}} \to \mathbb{R}$
- Submodular if for all $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{V} \setminus \{e\}$,




$$F(\mathcal{A}\cup\{e\}) - F(\mathcal{A}) \quad \geq \quad F(\mathcal{B}\cup\{e\}) - F(\mathcal{B})$$

i.e., diminishing returns

- Exact minimization in polynomial-time [Grotschel *et al.*, 1981].
- A quadratic set-function is submodular iff all its off-diagonal elements are non-positive [Bach, 2013].

## The Recipe:

- Subset-selection form:

$$\min_{\mathcal{S}\subseteq\mathcal{V}}\left\{f(\mathcal{S}) := \mathbb{1}_{\mathcal{S}}^T\mathbf{A}\mathbb{1}_{\mathcal{S}} + \mathbf{b}^T\mathbb{1}_{\mathcal{S}}\right\}$$

- Express $f(\mathcal{S})$ as a difference of submodular functions, $\mathbf{A} = \mathbf{A}_1 - \mathbf{A}_2$, where $\mathbf{A}_1 = \min\{\mathbf{A}, \mathbf{0}\}$ and $\mathbf{A}_2 = \min\{-\mathbf{A}, \mathbf{0}\}$
- Obtain representation

$$\min_{\mathcal{S}\subseteq\mathcal{V}}\left\{f(\mathcal{S}) := \underbrace{\mathbb{1}_{\mathcal{S}}^T\mathbf{A}_1\mathbb{1}_{\mathcal{S}} + \mathbf{b}^T\mathbb{1}_{\mathcal{S}}}_{g(\mathcal{S})} - \underbrace{\mathbb{1}_{\mathcal{S}}^T\mathbf{A}_2\mathbb{1}_{\mathcal{S}}}_{h(\mathcal{S})}\right\}$$

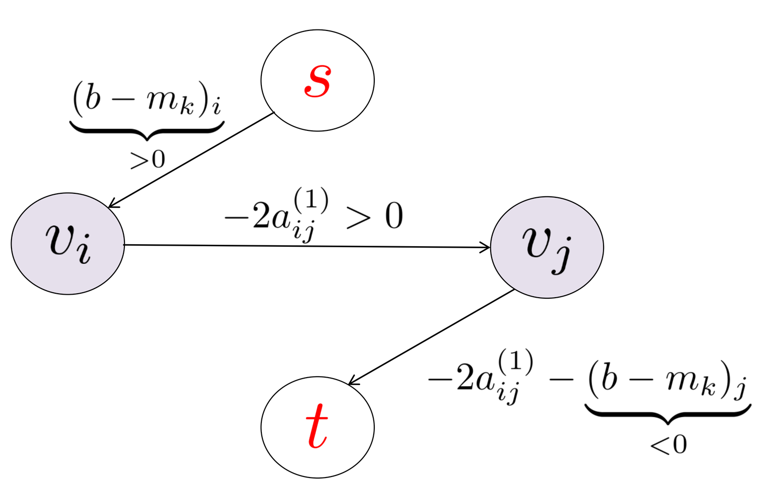
- Majorize $h(\mathcal{S})$ via modular function $m_k(\mathcal{S}) := \mathbf{m}_k^T\mathbb{1}_{\mathcal{S}}$ satisfying

$$(a)\ h(\mathcal{S}) \geq m_k(\mathcal{S}), \forall\, \mathcal{S}\subseteq\mathcal{V}, \qquad (b)\ h(\mathcal{S}_k) = m_k(\mathcal{S}_k)$$

- Obtain quadratic submodular minimization sub-problem

$$\min_{\mathcal{S}\subseteq\mathcal{V}}\left\{\phi_k(\mathcal{S}) := \mathbb{1}_{\mathcal{S}}^T\mathbf{A}_1\mathbb{1}_{\mathcal{S}} + (\mathbf{b}-\mathbf{m}_k)^T\mathbb{1}_{\mathcal{S}}\right\}$$

- Optimally solvable via Max-Flow [Kolmogorov-Zabih, 2004]




Max-Flow = Min-Cut
[Ford-Fulkerson, 1962]

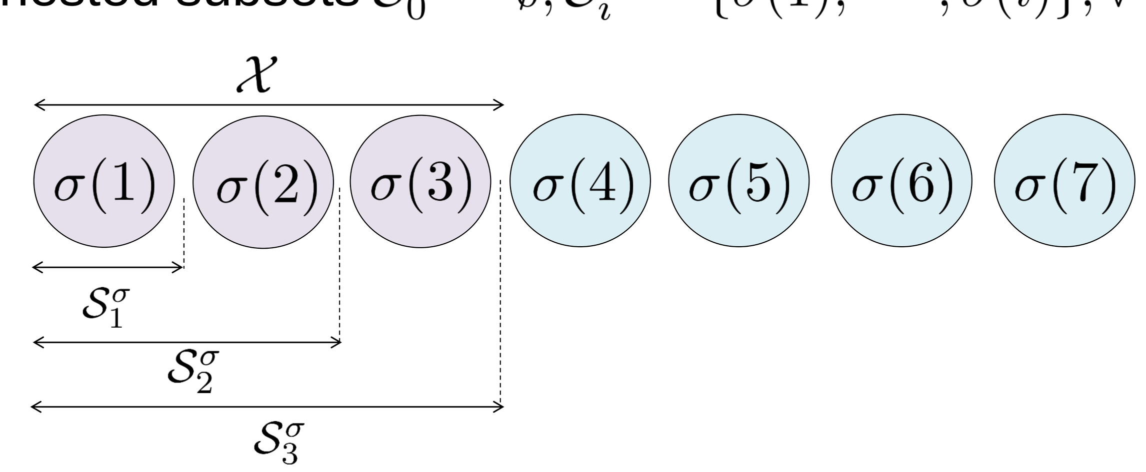
- Sequentially minimize series of global submodular upper bounds

## Construction of majorizer: [Narasimhan-Bilmes, 2005]

- **Key fact:** submodular functions possess sub-differential sets, [Fujishige, 2004]

$$\partial h(\mathcal{X}) = \{\mathbf{v}\in\mathbb{R}^n : h(\mathcal{Y}) \geq h(\mathcal{X}) + \mathbf{v}^T\mathbb{1}_{\mathcal{Y}} - \mathbf{v}^T\mathbb{1}_{\mathcal{X}}, \forall\, \mathcal{Y}\subseteq\mathcal{V}\}$$

- Compute sub-gradient of $\partial h(\mathcal{X})$ using algorithm of [Edmonds, 1970]
- Generate permutation $\boldsymbol{\sigma}\in[n]$ with entries $\sigma(i)\in\mathcal{X}, \forall\, i \leq |\mathcal{X}|$
- Generate nested subsets $\mathcal{S}_0^{\boldsymbol{\sigma}} = \emptyset, \mathcal{S}_i^{\boldsymbol{\sigma}} = \{\sigma(1), \cdots, \sigma(i)\}, \forall\, i\in[n]$




- Generate $\mathbf{v}_{\mathcal{X}}^{\boldsymbol{\sigma}}\in\mathbb{R}^n$ with entries $v_{\mathcal{X}}^{\boldsymbol{\sigma}}(\sigma(i)) = h(\mathcal{S}_i^{\boldsymbol{\sigma}}) - h(\mathcal{S}_{i-1}^{\boldsymbol{\sigma}}), \forall\, i\in[n]$

## The Algorithm:

- **Initialize:** $\mathcal{S}_0 \subseteq \mathcal{V}$ and generate permutation $\boldsymbol{\sigma}_0$ corresponding to $\mathcal{S}_0$, set $k = 0$
- **Repeat:**
- Compute sub-gradient $\mathbf{v}_{\mathcal{S}_k}^{\boldsymbol{\sigma}_k} \in \partial h(\mathcal{S}_k)$ via Edmond's procedure
- Use a Max-Flow algorithm to compute

$$\mathcal{S}_{k+1} \in \arg\min_{\mathcal{S}\subseteq\mathcal{V}}\ \{\phi_k(\mathcal{S}) := \mathbb{1}_{\mathcal{S}}^T\mathbf{A}_1\mathbb{1}_{\mathcal{S}} + (\mathbf{b} - \mathbf{v}_{\mathcal{S}_k}^{\boldsymbol{\sigma}_k})^T\mathbb{1}_{\mathcal{S}}\}$$

- Form random permutation $\boldsymbol{\sigma}_{k+1}$ corresponding to $\mathcal{S}_{k+1}$
- Set $k = k+1$
- **Until:** termination criterion is met
- **Properties:**
  - Successive approximation in {0,1} domain
  - Naturally respects {0,1} constraints
  - Monotonically reduces cost function

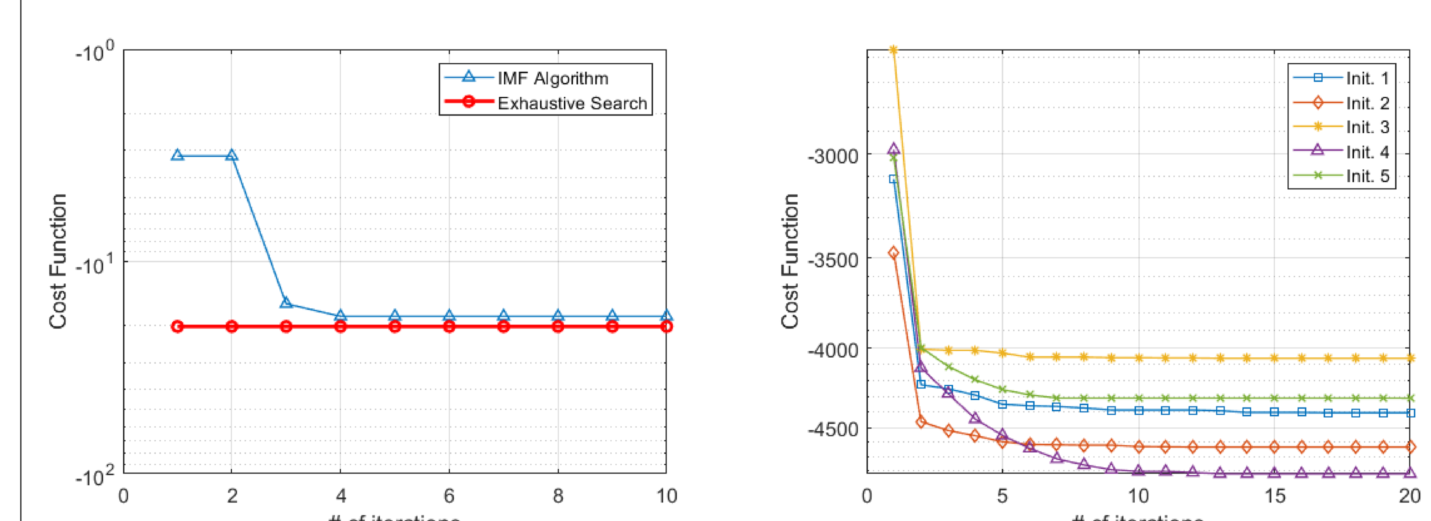
## Preliminary Results:

- Use Matlab's built in Max-Flow solver [Boykov-Kolmogorov, 2004]
- Mode computation in {0,1} undirected MRFs

$$P(x_1, x_2, \cdots, x_n) = \frac{1}{Z}\exp\Bigg(-\sum_{i\in[n]} b_i x_i - \sum_{(i,j)\in\mathcal{E}} a_{ij}x_ix_j\Bigg)$$

- Generate complete graph of pairwise-potentials, and randomly remove subset of edges

**Figure: Evolution of cost function with iterations for n = 15 and n = 1000**




**Near-optimal for small instances at low complexity**

**Scalable to large instances, dependent on choice of initialization**

## Conclusions:

- Approximate minimization of Boolean QP via iterative submodular approximation
- Future work – Tests on real world data, provide theoretical guarantees on performance.