# Parametric Frugal Sensing of Moving Average Power Spectra


**Aritra Konar and Nicholas D. Sidiropoulos**

University of Minnesota

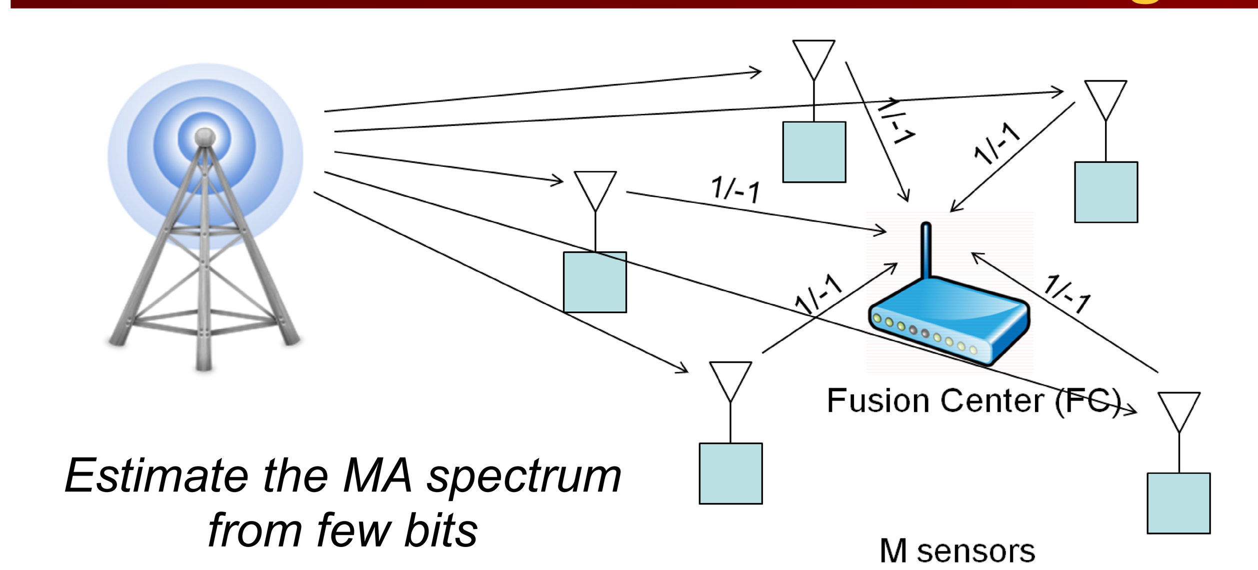
UNIVERSITY OF MINNESOTA


## Collaborative Moving Average Spectrum Estimation

- Crowdsourced spectrum sensing: Exploit spatial diversity in distributed sensors to avoid hidden termial problem, mitigate fading, and enhance spectrum estimation reliability
- Classical spectrum estimation techniques not suitable in distributed sensing scenarios
- Sending analog or finely quantized signal samples creates a heavy burden in terms of communication overhead and battery lifetime ⟹ send one (few) bits/sensor: Frugal Sensing [Mehanna *et al*., 2013]
- Assumption: primary signal admits a Moving Average (MA) parametrization (e.g., FIR pulse-shaping of digital communication signals)

***Challenge:*** *collaborative MA spectrum estimation using low-end sensors with limited communication capabilities*

## Frugal Sensing



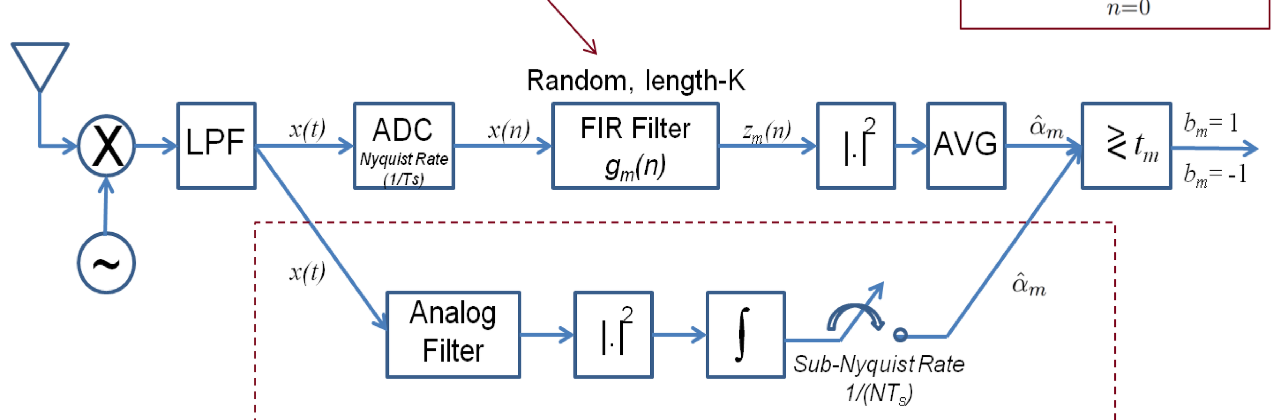

*Estimate the MA spectrum from few bits*

Complex PN - known at the FC

$$g_m(n) = \begin{cases} (1/\sqrt{2K})(\pm 1 \pm j) & \text{if } 0 \le n \le K-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha_m := \mathbb{E}[|z_m(n)|^2] \qquad \hat{\alpha}_m = \frac{1}{N}\sum_{n=0}^{N-1}|z_m(n)|^2$$

Random, length-K




*Equivalent analog measurement*

**MA Signal Model**

$$x(n) = \sum_{k=0}^{q} h(k)w(n-k)$$

MA parameters — Complex WGN

**Signal Autocorrelation**

$$r_x(k) = \begin{cases} \sum_{i=0}^{q-|k|} h^*(i)h(i+|k|) & : |k| \le q \\ 0 & : |k| > q \end{cases}$$

$$= \begin{cases} \mathbf{h}^H \mathbf{\Theta}_k^{(q+1)} \mathbf{h} & : |k| \le q \\ 0 & : |k| > q \end{cases}$$

**Power Measurement**

$$\hat{\alpha}_m = \mathbb{E}[|z_m|^2] = \mathbb{E}[|\mathbf{g}_m^H \mathbf{x}|^2] = \mathbf{g}_m^H \mathbf{R}_x \mathbf{g}_m \longrightarrow \text{Linear in the auto-correlation}$$

$$\mathbf{g}_m^H \mathbf{R}_x \mathbf{g}_m = \mathbf{g}_m^H \left( r_x(0)\mathbf{\Theta}_0^K + \sum_{k=1}^{\min(K-1,q)} (r_x(k)\mathbf{\Theta}_k^K + r_x^*(k)\mathbf{\Theta}_{-k}^K) \right) \mathbf{g}_m$$

$$= \underbrace{\mathbf{g}_m^H \mathbf{\Theta}_0^K \mathbf{g}_m}_{c_{m,0}} r_x(0) + \sum_{k=1}^{min(K-1,q)} \left( \underbrace{\mathbf{g}_m^H \mathbf{\Theta}_k^K \mathbf{g}_m}_{c_{m,k}} r_x(k) + \underbrace{\mathbf{g}_m^H \mathbf{\Theta}_{-k}^K \mathbf{g}_m}_{c_{m,-k}} r_x^*(k) \right)$$

$$= c_{m,0} r_x(0) + \sum_{k=1}^{\min(K-1,q)} (c_{m,k} r_x(k) + c_{m,-k} r_x^*(k))$$

$$= \mathbf{h}^H \left( c_{m,0}\mathbf{\Theta}_0^{(q+1)} + \sum_{k=1}^{min(K-1,q)} (c_{m,k}\mathbf{\Theta}_k^{(q+1)} + c_{m,-k}\mathbf{\Theta}_{-k}^{(q+1)}) \right) \mathbf{h}$$

$$= \mathbf{h}^H \mathbf{C}_m \mathbf{h} \longrightarrow \text{Quadratic in the MA parameters}$$

**1-Bit Measurement**

$$b_m = \text{sign}(\mathbf{h}^H \mathbf{C}_m \mathbf{h} - t_m)$$

Goal: Estimate $\mathbf{h}$ from $\{b_m\}_{m=1}^M$

## Quadratically Constrained Quadratic Programming (QCQP) Formulation

- Constraints
  - Assume single threshold: $t_m = t$
  - Define sets $\mathcal{M}_a := \{m : b_m = 1\}$, $\mathcal{M}_b := \{m : b_m = -1\}$, $|\mathcal{M}_a| + |\mathcal{M}_b| = M$
  - Received bits: $b_m = 1 \implies \mathbf{h}^H \mathbf{C}_m \mathbf{h} \ge t, \ \forall m \in \mathcal{M}_a$; $b_m = -1 \implies \mathbf{h}^H \mathbf{C}_m \mathbf{h} < t, \ \forall m \in \mathcal{M}_b$
- Cost function
  - Average signal power: $\mathbb{E}[|x(n)|^2] = r_x(0) = \mathbf{h}^H\mathbf{h} = \|\mathbf{h}\|_2^2$

- QCQP formulation

$$(P) \quad \min_{\mathbf{h} \in \mathbb{C}^{p+1}} \ \|\mathbf{h}\|_2^2 \quad \text{s.t.} \quad \mathbf{h}^H \mathbf{C}_m \mathbf{h} \ge t, \ m \in \mathcal{M}_a; \quad \mathbf{h}^H \mathbf{C}_m \mathbf{h} < t, \ m \in \mathcal{M}_b$$

*Non-convex problem, known to be NP-Hard in general*

MA power spectrum: $S_x(e^{j\omega}) = |\sum_{n=0}^{q} h(n)e^{-j\omega n}|^2$

## Semidefinite Programming (SDP) Relaxation and other approaches

**Equivalent Rank Constrained SDP formulation**

- Define $\mathbf{H} := \mathbf{h}\mathbf{h}^H$

$$\min_{\mathbf{H} \in \mathbb{C}^{(p+1)\times(p+1)}} \ \text{Trace}(\mathbf{H}) \quad \text{s.t.} \quad \text{Trace}(\mathbf{C}_m\mathbf{H}) \ge t, \ m \in \mathcal{M}_a; \quad \text{Trace}(\mathbf{C}_m\mathbf{H}) < t, \ m \in \mathcal{M}_b; \quad \mathbf{H} \succeq \mathbf{0}, \quad \text{rank}(\mathbf{H}) = 1$$

Rank Relaxation

**SDP Relaxation**

$$\min_{\mathbf{H} \in \mathbb{C}^{(p+1)\times(p+1)}} \ \text{Trace}(\mathbf{H}) \quad \text{s.t.} \quad \text{Trace}(\mathbf{C}_m\mathbf{H}) \ge t, \ m \in \mathcal{M}_a; \quad \text{Trace}(\mathbf{C}_m\mathbf{H}) < t, \ m \in \mathcal{M}_b; \quad \mathbf{H} \succeq \mathbf{0}$$

*Lagrangian bi-dual of (P)*

**Randomization Algorithm**

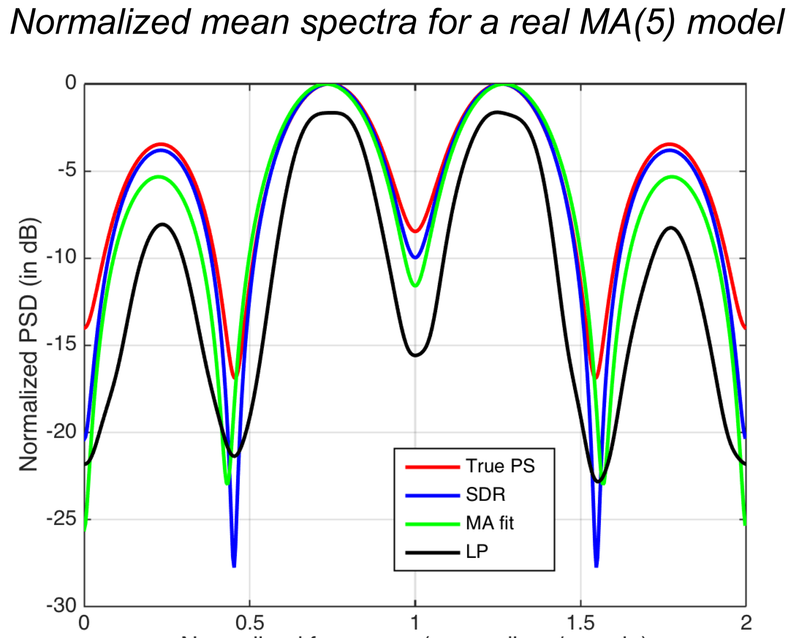
- If solution of SDP relaxation is rank-1, then global optimum achieved
- Randomization Approach
  - Scale principal component of SDP solution to be feasible for
  - Employ *Gaussian Randomization* to obtain feasible solution $(P)$
  - If randomization fails to produce a feasible solution,
    - Scale principal component/use Gaussian Randomization to obtain feasible solutions for $\mathcal{M}_a$ only
    - Justification: $\mathcal{M}_a$ is the activity detection set, MVDR interpretation
- Randomization algorithm fails to yield feasible solution in most cases (observed from simulations)
- Interestingly, fall-back procedure still yields good quality estimates in many cases

**Current State-of-the-Art**

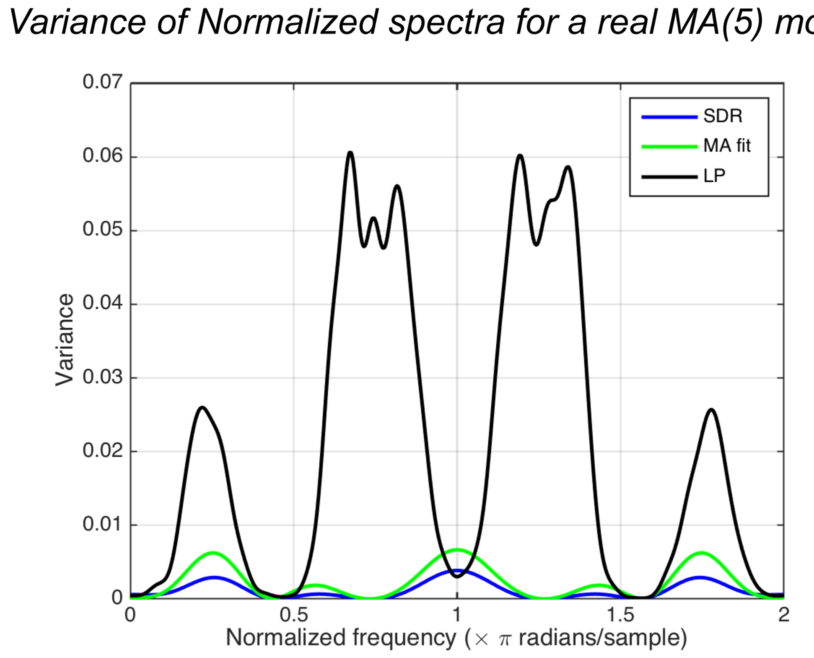
- Non parametric Linear Programming (LP) approach (Ref: O. Mehanna and N. D. Sidiropoulos, "Frugal Sensing: Wideband Power Spectrum Sensing from few bits", *IEEE TSP, May 2013*)
- Classical MA spectrum estimation tools (Ref: P. Stoica and R. L. Moses, "Spectral Analysis of Signals")
- SDP relaxation + Spectral Factorization. Solves *(P)* to global optimality in polynomial time! (Ref: A. Konar and N. D. Sidiropoulos, "Hidden Convexity in QCQP with Toeplitz-Hermitian Quadratics", *IEEE SPL, Oct. 2015*)

## Numerical Results

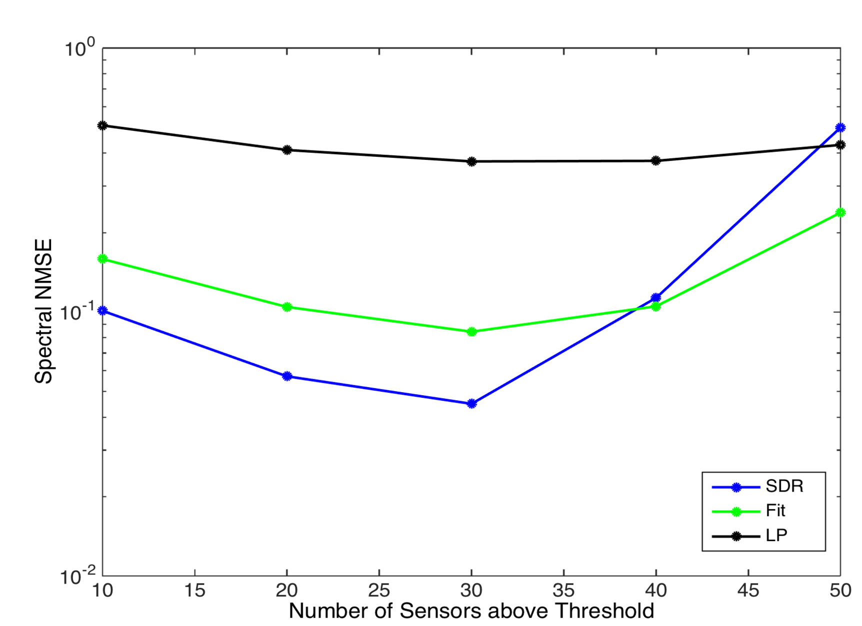
*Normalized mean spectra for a real MA(5) model*




*Variance of Normalized spectra for a real MA(5) model*




*Spectral NMSE vs number of sensors above threshold for MA(9) models*




## Conclusions

- Adequate wideband MA spectrum estimation possible from few bits.
- Exploiting underlying MA structure produces more accurate estimation results as compared to non - parametric approaches