Hidden Convexity in QCQP with Toeplitz-Hermitian Quadratics

Aritra Konar, *Student Member, IEEE*, and Nicholas D. Sidiropoulos, *Fellow, IEEE*

***Abstract*—Quadratically Constrained Quadratic Programming (QCQP) has a broad spectrum of applications in engineering. The general QCQP problem is NP–Hard. This article considers QCQP with Toeplitz-Hermitian quadratics, and shows that it possesses hidden convexity: it can always be solved in polynomial-time via Semidefinite Relaxation followed by spectral factorization. Furthermore, if the matrices are circulant, then the QCQP can be equivalently reformulated as a linear program, which can be solved very efficiently. An application to parametric power spectrum sensing from binary measurements is included to illustrate the results.**

***Index Terms*—Circulant-Toeplitz QCQP, distributed spectrum sensing, linear programming, moving-average processes, Toeplitz-Hermitian QCQP, semi-definite relaxation, spectral factorization.**

I. Introduction

QUADRATICALLY Constrained Quadratic Programming (QCQP) is an important class of optimization problems that arise in various science and engineering fields, ranging from wireless communications and networking, (e.g., $\{\pm 1\}$ multiuser detection [1], multicast beamforming [2]–[4], and the MAXCUT problem [5]), to radar (e.g., robust adaptive radar detection [6], and optimum coded waveform design [7]), and signal processing (e.g., parametric model-based power spectrum sensing [8]). A QCQP can be expressed as

$$\min_{\mathbf{x}\in\mathbb{C}^n} \quad \mathbf{x}^H\mathbf{A}_0\mathbf{x} \tag{1a}$$
$$\text{s.t.} \quad \mathbf{x}^H\mathbf{A}_m\mathbf{x} \le u_m, \forall m \in \mathcal{M} \tag{1b}$$

where $\mathbf{A}_0 \succeq 0$, i.e., positive semidefinite, $\mathbf{A}_m \in \mathbb{C}^{n\times n}$ are Hermitian matrices $\forall m \in \mathcal{M} = \{1, 2, \cdots, M\}$, while $\{u_m\}_{m=1}^M$ are real numbers. For the special case $\mathbf{A}_m \succeq 0 \,\forall m \in \mathcal{M}$, the problem is convex and can be efficiently solved to global optimality using interior point algorithms [9]. However, the general case of the QCQP (1) (where the quadratic constraints (1b) involve negative semidefinite or indefinite matrices) is non-convex and is known to be NP–Hard [10]. In fact, for an arbitrary instance of (1), even establishing (in)feasibility is NP–Hard. Only in certain cases, involving a small number of non-convex constraints (e.g., see [11]–[20]) or special problem structure, or both (e.g., see [21]–[27]) can (1) be solved to global optimality in polynomial time. We say that such special instances possess *hidden convexity*.

A popular polynomial-time approximation strategy for obtaining sub-optimal solutions to (1) is that of Semidefinite Relaxation (SDR) [28]. Defining $\mathbf{X} = \mathbf{x}\mathbf{x}^H$ and utilising the cyclic property of the trace operator, (1) can be equivalently recast as the following rank constrained Semidefinite Programming (SDP) problem.

$$\min_{\mathbf{X}\in\mathbb{C}^{n\times n}} \quad \text{Trace}(\mathbf{A}_0\mathbf{X}) \tag{2a}$$
$$\text{s.t.} \quad \text{Trace}(\mathbf{A}_m\mathbf{X}) \le u_m, \forall m \in \mathcal{M} \tag{2b}$$
$$\mathbf{X} \succeq \mathbf{0}, \tag{2c}$$
$$\text{Rank}(\mathbf{X}) = 1 \tag{2d}$$

Upon dropping the non-convex rank constraint, we obtain the following rank-relaxed SDP problem

$$\min_{\mathbf{X}\in\mathbb{C}^{n\times n}} \quad \text{Trace}(\mathbf{A}_0\mathbf{X}) \tag{3a}$$
$$\text{s.t.} \quad \text{Trace}(\mathbf{A}_m\mathbf{X}) \le u_m, \forall m \in \mathcal{M} \tag{3b}$$
$$\mathbf{X} \succeq \mathbf{0} \tag{3c}$$

whose solution yields a lower bound on the optimal value of the cost function of (1). Note that (3) is the Lagrangian bi-dual of (1), and can be solved efficiently using modern interior point methods, at a worst case computational complexity of $\mathcal{O}(n^{6.5})$ [29]. If the optimal solution of (3) is rank-1, then its principal component is the globally optimal solution for (1). However, solving (3) does not solve the original NP–Hard problem (1) in general, i.e., the rank of the optimal solution of (3) is generally higher than 1.

In this article, we consider the special case of the non-convex QCQP (1) where all the matrices are Toeplitz-Hermitian. No additional structure is assumed, except $\mathbf{A}_0 \succeq 0$ (so that we always have a valid minimization problem). It is shown that for this special case of (1),

1) (In)feasibility can always be established in polynomial-time; and
2) If the problem is feasible, then it can be solved to global optimality in polynomial-time too.

Our proof uses the Toeplitz structure of the matrices to show the tightness of SDR, although simply solving SDR for this special case of the QCQP problem does not return a rank-1 solution in general. Instead, we use a relaxed SDP formulation for (1) based on representation of finite autocorrelation sequences (FAS) via Linear Matrix Inequalities (LMIs) to show the existence of a rank-1 solution, which is also shown to be equivalent to SDR. A rank-reduction technique based on spectral factorization is used to convert the higher rank solution of SDR into a feasible rank-1 solution with the same cost. The proof of

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A. Konar is with the Department of Electrical and Computer Engineering, University of Minnesota Digital Technology Center, Minneapolis MN 55455 USA (e-mail: konar006@umn.edu).

N. D. Sidiropoulos holds an ADC Endowed Chair in Digital Technology, Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455 USA (e-mail: nikos@umn.edu)

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tightness does not depend on the number of constraints M. The implication is that non-convex QCQP with Toeplitz-Hermitian quadratics is not NP–Hard, but in fact exactly solvable in polynomial time.

To the best of our knowledge, our work is the first to show that *any* non-convex QCQP with Toeplitz-Hermitian quadratics can be solved optimally. Special cases have been previously considered in [30], [31], but none of them settled the *general* non-convex Toeplitz-Hermitian QCQP problem. Toeplitz quadratic minimization subject to Toeplitz *equality* constraints was considered in [30, p.30] (each equality corresponds to a pair of inequalities with the same Toeplitz-Hermitian matrix). Another special case was investigated in [31], which considered positive-semidefinite Toeplitz-Hermitian quadratics and a special QCQP structure arising in multi-group multicast beamforming. In [31], the proof of existence of an optimal rank-1 solution uses the Caratheodory parametrization of a covariance matrix [32, p. 181], which is only valid for *positive-semidefinite* Toeplitz matrices. Our work can be considered as an extension of this result, since we prove the existence of an optimal rank-1 solution for *indefinite* Toeplitz matrices. In [31], the optimal solution is obtained from the solution of the SDP relaxation based on the LMI representation of FAs. We show that this problem is equivalent to SDR, which is cheaper computationally, and demonstrate how an optimal rank-1 solution can be obtained, which also solves the original non-convex QCQP problem.

When all matrices are circulant (a special class of Toeplitz matrices), we further show that the QCQP problem (1) can be equivalently reformulated as a Linear Programming (LP) problem, which can again be solved to global optimality in polynomial-time, at a far lower computational complexity compared to the SDR approach.

II. QCQPs with Toeplitz Quadratic Forms

Consider a special case of (1) where the Hermitian matrices $\{\mathbf{A}_m\}_{m=0}^M$ are also Toeplitz. Each $\mathbf{A}_m$ can then be written as

$$\mathbf{A}_m = \sum_{k=-p}^{p} a_{m,k}\Theta_k \tag{4}$$

where $p = n-1$, Θ_k is an $n \times n$ elementary Toeplitz matrix with ones on the k^{th} diagonal and zeros elsewhere ($k = 0 \leftrightarrow$ main diagonal, $k > 0 \leftrightarrow$ super-diagonals, and $k < 0 \leftrightarrow$ sub-diagonals), while $a_{m,k}$ represents the entries along the k^{th} diagonal, i.e., $A_m(i,j) = a_{m,k} \forall j - i = k \in \mathcal{K}$, where $\mathcal{K} = \{-p, \cdots, 0, \cdots, p\}$. Note that due to the Hermitian property, $a_{m,k} = a^*_{m,-k} \forall k \in \mathcal{K}$. Using (10), we can express each quadratic term in (1) as $\mathbf{x}^H\mathbf{A}_m\mathbf{x} = \mathrm{Re}(\mathbf{a}_m^T\mathbf{r})$, where $\mathbf{a}_m = [a_{m,0}, 2a_{m,1} \cdots, 2a_{m,p}]^T \in \mathbb{C}^n$, $\mathbf{r} = [r(0), \cdots, r(p)]^T \in \mathbb{C}^n$, $r(k) = \mathrm{Trace}(\Theta_k\mathbf{X})$, and $\mathbf{X} = \mathbf{x}\mathbf{x}^H$. Overall, (1) can be expressed as

$$\begin{aligned} \min_{\mathbf{X},\mathbf{r}}.\quad & \mathrm{Re}(\mathbf{a}_0^T\mathbf{r}) && (5a)\\ \text{s.t.}\quad & \mathrm{Re}(\mathbf{a}_m^T\mathbf{r}) \le u_m, \forall m \in \mathcal{M} && (5b)\\ & r(k) = \mathrm{Trace}(\Theta_k\mathbf{X}), \forall\, k \in \mathcal{K}_+ && (5c)\\ & \mathbf{X} \succeq \mathbf{0}, && (5d)\\ & \mathrm{Rank}(\mathbf{X}) = 1, && (5e) \end{aligned}$$

where $\mathcal{K} \supseteq \mathcal{K}_+ = \{0, \cdots, p\}$. Note that (5c) and (5d) $\Rightarrow r^*(k) = r(-k)$. Upon dropping the rank constraint, we obtain the following convex SDP relaxation.

$$\begin{aligned} \min_{\mathbf{X},\mathbf{r}}.\quad & \mathrm{Re}(\mathbf{a}_0^T\mathbf{r}) && (6a)\\ \text{s.t.}\quad & \mathrm{Re}(\mathbf{a}_m^T\mathbf{r}) \le u_m, \forall m \in \mathcal{M} && (6b)\\ & r(k) = \mathrm{Trace}(\Theta_k\mathbf{X}), \forall k \in \mathcal{K}_+ && (6c)\\ & \mathbf{X} \succeq \mathbf{0} && (6d) \end{aligned}$$

Claim 1: For Toeplitz $\{\mathbf{A}_m\}_{m=0}^M$, problems (3) and (6) are equivalent.

Proof: For any $\mathbf{X}, \mathbf{r}$ satisfying (6c) and (6d)

$$\begin{aligned} \mathrm{Re}(\mathbf{a}_m^T\mathbf{r}) &= a_{m,0}r(0) + 2\sum_{k=1}^{p}\mathrm{Re}(a_{m,k}r(k)) && (7a)\\ &= \sum_{k=-p}^{p} a_{m,k}r(k) && (7b)\\ &= \sum_{k=-p}^{p} a_{m,k}\mathrm{Trace}(\Theta_k\mathbf{X}) && (7c)\\ &= \mathrm{Trace}((\sum_{k=-p}^{p} a_{m,k}\Theta_k)\mathbf{X}) && (7d)\\ &= \mathrm{Trace}(\mathbf{A}_m\mathbf{X}) && (7e) \end{aligned}$$

therefore we may replace every instance of $\mathrm{Re}(\mathbf{a}_m^T\mathbf{r})$ in (6) (including the cost function $\leftrightarrow m = 0$) with $\mathrm{Trace}(\mathbf{A}_m\mathbf{X})$. Then it becomes evident that $\mathbf{r}$ is completely determined by $\mathbf{X}$ via (6c). Thus we may drop (6c) and simply compute $\mathbf{r}$ from the optimal $\mathbf{X}$. What remains is precisely (3). □

We next show that feasibility of any instance of (1) can always be checked in polynomial time by checking the feasibility of (6). Furthermore, if feasibility of (1) is established, then the optimal solution of (6) can always be transformed into a globally optimal solution of (1). Since (6) is equivalent to (3), a solution to (1) can also be obtained from a solution of (3), as we will soon show. Although solving (3) is more computationally efficient compared to (6) (since it has fewer variables and constraints), it is more convenient to establish the proof of the following claims by considering (6).

Claim 2: For Toeplitz $\{\mathbf{A}_m\}_{m=0}^M$, (in)feasibility of (6) is equivalent to (in)feasibility of (1). Furthermore, if (1) is feasible, then it can be solved to global optimality in polynomial time.

Proof: Taken together, constraints (5c), (5d), (5e) constitute the LMI parametrization of the autocorrelation sequence $r(k)$ of an MA process of order p, and it has been shown in [33, Appendix A] that (5e) is redundant, in the sense that the set of feasible $(\mathbf{X}, \mathbf{r})$ defined by (5c), (5d), and (5e) is the same as that defined by (5c) and (5d) only. If a solution $\hat{\mathbf{X}}, \hat{\mathbf{r}}$ of (6) has $\mathrm{Rank}(\hat{\mathbf{X}}) > 1$, then there exists a rank-1 matrix $\bar{\mathbf{X}}$ which defines the same sequence $\hat{\mathbf{r}}$, and such a rank-1 matrix can be obtained by determining a spectral factor $\bar{\mathbf{X}}$ of $\hat{\mathbf{r}}$ using a spectral factorization algorithm (e.g., see [34]) and setting $\bar{\mathbf{X}} = \bar{\mathbf{x}}\bar{\mathbf{x}}^H$. Spectral factorization is non-unique, but we only need one (e.g., the minimum phase) factor. □

The implication is that for the special case of (1) considered here, with Toeplitz-Hermitian quadratic forms, the problem is not NP–Hard (as is general QCQP with non-convex Hermitian quadratic forms), but is in fact exactly solvable in polynomial-time using convex programming, followed by a simple post-processing step.

A globally optimal solution of (1) can also be obtained by solving (1) first, followed by defining the autocorrelation sequence $r(k) = \mathrm{Trace}(\Theta_k\mathbf{X}_{\mathrm{opt}}) \forall k \in \mathcal{K}$, where $\mathbf{X}_{\mathrm{opt}}$ denotes the optimal solution of (3). Since solving (3) is equivalent to solving (6), determining a spectral factor of $\{r(k)\}_{k=-p}^{p}$ will yield the optimal solution to (1). This is the preferred approach since solving (3) is more computationally efficient compared to (6).

Thus, for QCQPs with non-convex Toeplitz-Hermitian quadratic forms, the solution of SDR (which is not rank-1 in general), can always be converted into an optimal rank-1 solution via spectral factorization; SDR is tight.

III. QCQPs with Circulant Quadratic Forms

We now consider a more special case of (1) where the matrices $\{\mathbf{A}_m\}_{m=0}^M$ are circulant. Although circulant matrices are a subset of the set of Toeplitz matrices, and hence the results of the previous section apply, we show that by exploiting the circulant structure, the QCQP problem (1) can be equivalently reformulated as a LP problem which can again be solved to global optimality, at a lower complexity cost as compared to solving the SDP (3). Circulant matrices are diagonalized by the DFT matrix, i.e., we can write $\mathbf{A}_m = \mathbf{F}\Lambda_m\mathbf{F}^H$, where $\mathbf{F} \in \mathbb{C}^{n\times n}$ is the unitary DFT matrix

$$\mathbf{F} = \frac{1}{\sqrt{n}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_n & w_n^2 & \cdots & w_n^{n-1} \\ 1 & w_n^2 & w_n^4 & \cdots & w_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w_n^{n-1} & w_n^{2(n-1)} & \cdots & w_n^{(n-1)(n-1)} \end{bmatrix} \tag{8}$$

where $w_n = e^{-j\frac{2\pi}{n}}$ is the nth root of unity, and Λ_m is a diagonal matrix of the eigenvalues of $\mathbf{A}_m$ obtained by taking the discrete Fourier Transform (DFT) of the first row of $\mathbf{A}_m$. Hence, each quadratic term in (1b) can be expressed as

$$\mathbf{x}^H\mathbf{A}_m\mathbf{x} = \mathbf{x}^H\mathbf{F}\boldsymbol{\Lambda}_m\mathbf{F}^H\mathbf{x} \tag{9a}$$
$$= \mathbf{y}^H\boldsymbol{\Lambda}_m\mathbf{y} \tag{9b}$$
$$= \sum_{i=1}^{n}\lambda_m(i)|y(i)|^2 \tag{9c}$$

with obvious notation. Define $z(i) = |y(i)|^2, \forall i \in \{1,\cdots,n\}$, $\mathbf{z} = [z(1),\cdots,z(n)]^T$, and $\boldsymbol{\lambda}_m = [\lambda_m(1),\cdots,\lambda_m(n)]^T$. Then, we have

$$\mathbf{x}^H\mathbf{A}_m\mathbf{x} = \boldsymbol{\lambda}_m^T\mathbf{z}, \forall m \in \mathcal{M} \tag{10}$$

Similarly for the objective, we have $\mathbf{x}^H\mathbf{A}_0\mathbf{x} = \boldsymbol{\lambda}_0^T\mathbf{z}$. Putting everything together, we obtain the following formulation

$$\min_{\mathbf{z}\in\mathbb{R}^n} \quad \boldsymbol{\lambda}_0^T\mathbf{z} \tag{11a}$$
$$\text{s.t.} \quad \boldsymbol{\lambda}_m^T\mathbf{z} \le u_m, \forall m \in \mathcal{M} \tag{11b}$$
$$\mathbf{z} \succeq 0 \tag{11c}$$

which is a LP problem in $\mathbf{z}$. Thus, by exploiting the fact that all circulant matrices are diagonalized by the *same* eigen-basis, we can equivalently reformulate the non-convex QCQP problem (1) as the LP problem (11), which can be solved to global optimality very efficiently. If $\mathbf{z}_{\text{opt}}$ is the optimal solution of (11), then we define $\mathbf{y}_{\text{opt}}$ as $y_{opt}(i) = \sqrt{z_{opt}(i)}, \forall i \in \{1,\cdots,n\}$. Since $\mathbf{F}$ is unitary, an optimal solution $\mathbf{x}_{\text{opt}}$ for (1) can be obtained as $\mathbf{x}_{\text{opt}} = \mathbf{F}\mathbf{y}_{\text{opt}}$. Again, the optimal solution is not unique since, from the definition of $\mathbf{z}$, all phase information about $\mathbf{y}$ is irrelevant.

IV. Numerical Results

In order to illustrate our claims, we carried out the following simulation experiments in MATLAB on a 64-bit desktop with 8 GB RAM and a 3.40 GHz Intel CORE i7 processor. YALMIP was chosen as the modeling language and the MOSEK solver was used to solve the optimization problems.

TABLE I
Results using SDR + Spectral Factorisation for Toeplitz Quadratic QCQPs

M	20	40	60	80	100
Avg. Loss (dB $\times 10^{-5}$)	6.37	6.92	6.06	5.97	9.53
Execution Time (secs)	0.114	0.154	0.192	0.231	0.268

In our first experiment, we consider a problem with $n = 10$ complex dimensions, $M \in \{20, 40, 60, 80, 100\}$, and all the matrices are Toeplitz-Hermitian. The following procedure was used to create each instance of the QCQP problem (1). Generating the Toeplitz-Hermitian constraint matrices $\{\mathbf{A}_m\}_{m=1}^M$ only requires specification of the first row of each matrix. Every such row is drawn randomly and independently from a complex, circularly symmetric Gaussian distribution with zero mean and covariance matrix equal to identity. The Hermitian part of each matrix was finally taken to obtain $\mathbf{A}_m$. An initial point $\mathbf{p}$ was randomly generated, while the each of the values $\{u_m\}_{m=1}^M$ were sampled randomly from a Gaussian distribution $u_m \sim \mathcal{N}(\mathbf{p}^H\mathbf{A}_m\mathbf{p}, 1)$. If $\mathbf{p}^H\mathbf{A}_m\mathbf{p} > u_m$, then both sides of the inequality are multiplied by -1 to obtain $\le$ inequalities. In order to generate $\mathbf{A}_0$, a Gaussian random vector was randomly drawn from the previous distribution, and its one-sided deterministic autocorrelation sequence was calculated, which was then used to specify the first row of $\mathbf{A}_0$. This technique ensures that $\mathbf{A}_0$ is Toeplitz-Hermitian, and also positive semi-definite. In order to solve (1), the SDR problem (1) was solved first, followed by a spectral factorization step to obtain a globally optimal solution of (1). The results, depicted in Table I, were obtained after averaging over 1000 Monte-Carlo trials.

The table reports the average difference between the cost of the rank-1 solution obtained from SDR followed by spectral factorization and the lower bound obtained from the (higher-rank) solution of SDR alone; plus the average execution time of the algorithm. It was observed that, in all instances, a feasible solution was obtained, at a very modest computational effort, which is extremely close to the SDR lower bound.

In our second experiment, we consider the case of the QCQP problem (1) where all the matrices are Circulant-Hermitian. The number of complex dimensions was set to be $n = 20$ and the constraints $M \in \{25, 50, 75, 100, 125\}$. In order to generate each of the constraint matrices $\{\mathbf{A}_m\}_{m=1}^M$, a random vector of eigen-values was independently sampled from a uniform distribution in the interval $[-10, 10]$. Upon forming a diagonal matrix of the eigen-values, followed by pre- and post-multiplication by the unitary DFT matrix [as defined in (8)], we obtain $\mathbf{A}_m$. The right hand sides $\{u_m\}_{m=1}^M$ were generated and the sign of each inequality was fixed in the same manner as described in our previous experiment. $\mathbf{A}_0$ was also synthesized in a similar fashion as each $\{\mathbf{A}_m\}_{m=1}^M$, except that the vector of eigen-values was drawn randomly from a uniform distribution on the interval [0,20], in order to ensure positive semi-definiteness. Each problem instance was solved by both the LP approach and SDR followed by spectral factorization. The results are summarized in Tables II and III.

Each table reports the average loss incurred in the cost function when the cost of the solutions obtained via the respective methods are compared to the SDR lower bound, along with the average execution times. From the tables, it is seen that the obtained solutions from both methods are indeed the globally optimal solutions of the non-convex QCQP problem (1), as evidenced by the very low average loss. As expected, the LP approach is considerably faster as compared to the SDR

TABLE II
RESULTS USING SDR + SPECTRAL FACTORISATION FOR CIRCULANT QUADRATIC QCQPS

M	25	50	75	100	125
Avg. Loss (dB $\times 10^{-5}$)	71.3	6.32	8.28	2.32	3.63
Execution Time (secs)	0.228	0.274	0.327	0.386	0.451

TABLE III
RESULTS USING LINEAR PROGRAMMING FOR CIRCULANT QUADRATIC QCQPS

M	25	50	75	100	125
Avg. Loss (dB $\times 10^{-6}$)	0.28	1.22	1.42	1.24	1.22
Execution Time (secs)	0.059	0.098	0.138	0.171	0.214

method. In addition, it was empirically observed that the solution of the former method was more accurate as compared to that of the latter method, in the sense that the average loss was typically an order of magnitude smaller. Overall, this illustrates the benefit of using the LP approach over the SDR method for solving non-convex QCQP problems with Circulant-Hermitian quadratic forms.

Parametric Frugal Sensing of Moving Average Power Spectra. We next consider an application to a distributed spectrum sensing problem originally formulated as a nonconvex QCQP in [8], and solved therein using approximation algorithms, in light of its apparent non-convexity. Using the system model defined in [8], the problem of estimating a Moving Average (MA) power spectrum from quantized power measurements can be cast as non-convex QCQP with two-sided constraints involving Toeplitz-Hermitian quadratics.

We considered a scenario where the signal was generated by pulse shaping streams of symbols drawn independently from a quadrature phase shift keying (QPSK) constellation using a discrete-time raised cosine filter spanning a window of 8 symbols, at 4 samples per symbol and a roll-off factor of 0.75. A total of $M = 150$ sensors were used for the spectrum sensing operation, and the length of the impulse response of the broadband pseudo-noise filters was set to be $K = 50$ (see [8] for the experimental setup–we omit many details here due to space limitations, but our experiments are reproducible if one also consults [8]). It was assumed that the true model order is known at the fusion center. We used SDR followed by spectral factorization to solve the problem optimally, and used its performance to benchmark the FPP-SCA algorithm proposed in [8]. The latter algorithm was initialized by the procedure described in [8], and run until the cost function did not improve more than 10^{-4} in the last 10 iterations. When this required > 40 iterations, FPP-SCA was re-initialized from a different starting point, up to a maximum of 5 such re-initializations. Upon obtaining a solution, the magnitude square of its DTFT is used to obtain a spectral estimate. We used the Normalized Mean Square Error (NMSE) as performance criterion, defined as $\mathbf{NMSE} = \mathbb{E}[\frac{\|S_x - \hat{S}_x\|^2}{\|S_x\|^2}]$, where S_x is the true spectrum and $\hat{S}_x$ is the estimated spectrum, with both spectra normalized by the respective peak values. The expectation is taken with respect to the randomness in the primary signal and the impulse responses of the wideband FIR filters employed by each sensor. The threshold t was tuned in order to vary the number of sensors M_a reporting above threshold, and the results were averaged over 400 Monte-Carlo trials for each value of M_a. It was observed that SDR followed by spectral factorization exhibited lower spectral NMSE as compared to the FPP-SCA algorithm - the gap is small for M_a up to 80 but increases sharply after that and is almost an order of magnitude

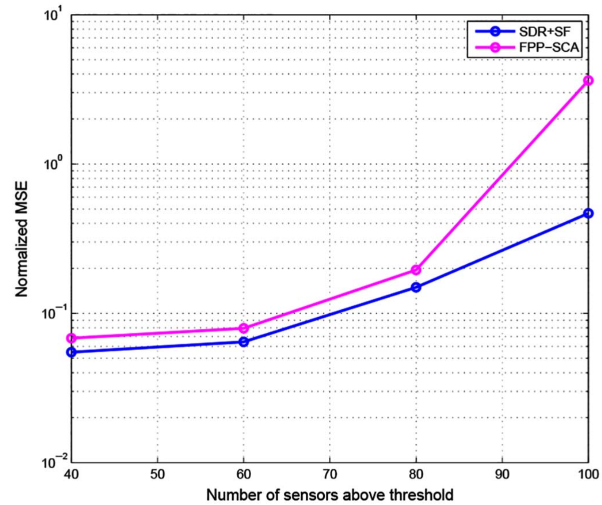

Fig. 1. NMSE versus M_a.

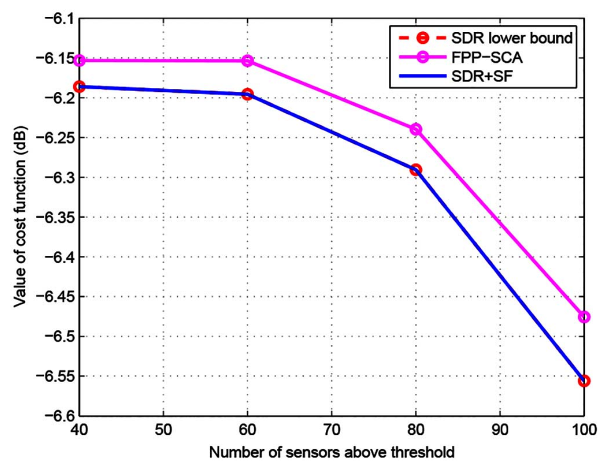

Fig. 2. Cost function versus M_a.

at $M_a = 100$. See Fig. 1. We note that while NMSE captures estimation performance, it only indirectly reflects optimization performance. From the optimization point of view, the cost of the solution obtained from spectral factorization always matches the SDR lower bound, see Fig. 2, implying that the algorithm is indeed successful in solving the non-convex QCQP problem to global optimality. Although FPP-SCA is what one would call ‘engineering approximation’, the cost of the FPP-SCA solution was on average only $\sim 0.03 - 0.08$ dB away from the globally optimal cost - see Fig. 2. A feasible solution was also identified in every trial, without any restarts.

V. CONCLUSIONS

We considered QCQPs with Toeplitz-Hermitian quadratics and proved that they are exactly solvable in polynomial-time via SDP relaxation followed by spectral factorization. For circulant-Hermitian quadratics, it was shown that the QCQP can be reformulated as LP, which can be solved very efficiently. Numerical experiments illustrated the main claims. The result was applied to a parametric model-based power spectrum sensing application, where the problem was solved to global optimality and used to benchmark the performance of the FPP-SCA algorithm, which is an iterative approximation technique for general non-convex QCQPs. Simulations indicate that the latter algorithm performs unexpectedly well in this context.

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